SPECTRAL CONDITION FOR HAMILTONICITY OF A GRAPH

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Let $G = (V(G), E(G))$ be a simple undirected connected graph of order n and let $A = A(G)$ be its adjacency matrix. The largest eigenvalue of A, denoted by $\rho(G)$, is called *the spectral radius* of the graph G. According to the Perron-Frobenius theory of matrices [1] the spectral radius $\rho(G)$ is a positive real root of multiplicity 1 of the characteristic polynomial $\det(xE_n - A)$ and there exists a unique positive unit eigenvector corresponding to $\rho(G)$, called the *Perron* vector. Let $d_i = \deg_G(v_i)$ be the degree of any vertex $v_i \in V$ of the graph G and $(d_1, d_2, ..., d_n)$ be the degree sequence of the graph G, where $d_1 \leq d_2 \leq ... \leq d_n$. Then $d_1 = \delta$ is called *the minimum degree.*

The union of two graphs G and H is the graph $G \cup H$ with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. If graphs G and H are disjoint, then we call their union *a disjoint union* and denote it by $G + H$. The union of k disjoint copies of a graph G is denoted by kG. *The join* of two disjoint graphs G and H, denoted by $G \vee H$, is obtained from $G + H$ by joining each vertex of G to each vertex of H.

A cycle or path passing through all the vertices of a graph is called *Hamiltonian.* A graph G, containing Hamiltonian cycle or path, is called *Hamiltonian* or *traceable* correspondingly. It is known that the problem of deciding whether a given graph is Hamiltonian or traceable is NP-complete. Recently, spectral theory of graphs has been applied to this problem. In particular, the following Brualdi-Solheid-Turan type problem [2] has been intensively studied:

Problem. *For a given graph F, what is the maximum spectral radius of a graph G on n vertices without a subgraph isomorphic to F?*

This problem has been considered for the cases where F is a clique, an even or odd path (cycle) of the given length and a Hamiltonian path (cycle) ([3–6]). For example, sufficient spectral conditions for the existence of Hamiltonian paths and cycles are studied. Fiedler, Nikiforov gave tight sufficient conditions for the existence of Hamiltonian paths and cycles in terms of the spectral radius of graphs or the complements of graphs [3]. Lu et al. [4] studied sufficient conditions for Hamiltonian paths in connected graphs and Hamiltonian cycles in bipartite graphs in terms of the spectral radius of a graph. Some other spectral conditions for Hamiltonian paths and cycles in graphs have been given in [7–8]. In 2014 [9] the sufficient condition of traceability of the graph was found in terms of the spectral radius:

Theorem 1 [9]. *Let G be a graph on $n \geq 4$ vertices with $\delta \geq 1$. If $\rho(G) > n - 3$, then G contains a Hamiltonian path unless $G \in \{K_1 \vee (K_{n-3} + 2K_1), K_2 \vee 4K_1, K_1 \vee (K_{1}, 3 + K_1)\}$.*

In this work the following sufficient condition for the Hamiltonicity of a given graph in terms of its spectral radius has been obtained.

Theorem 2. *Let G be a simple connected graph on $n > 8$ vertices with $\delta \geq 2$. If $\rho(G) \geq n-3$, then the graph G is Hamiltonian unless $G \in \{5K_1 \vee K_4, K_3 \vee (K_{1,4} + K_1), K_2 \vee (K_{n-4} + 2K_1)\}$.*

The proof of the theorem is based on the following well-known facts.

Lemma 1 [10]. *Let G be a simple graph with n vertices and m edges and δ be the minimum degree of vertices of G. Then its spectral radius $\rho(G)$ satisfies the inequality:*

$$\rho(G) \leq \frac{\delta - 1 + \sqrt{(\delta + 1)^2 + 4(2m - \delta n)}}{2}.$$

Lemma 2 [10]. $f(x) = x - 1 + \sqrt{(x + 1)^2 + 4(2m - xn)}$ *is a decreasing function of x on the interval $[1; n - 1]$, where $n - 1 \leq m \leq n(n - 1)/2$ and $2m \geq xn$.*

From lemmas 1, 2, and the conditions of the theorem one can receive the following inequality:

$$n^2 - 5n + 10 \leq 2m. \tag{1}$$

Let us assume that the graph G is not Hamiltonian. Then according to the Chvatal theorem [11] there exists a number $k \in \mathbb{N}$, such that $d_k \leq k < n/2$ and $d_{n-k} \leq n-k-1$ for the degree sequence of the graph G: $\delta = d_1 \leq d_2 \leq \ldots \leq d_n$. Therefore:

$$2m = \sum_{i=1}^{n} d_i \leq k \cdot k + (n-2k)(n-k-1) + k(n-1) = n^2 + 3k^2 + k - 2kn - n. \qquad (2)$$

and one can show that $k \in \{1, 2, \ldots, 6\}$. However, from the condition $\delta \geq 2$ it follows, that $k \neq 1$.

In the case $k = 2$ the inequality (2) gives the upper bound: $2m \leq n^2 - 5n + 14$. Thus:

$$\frac{n(n-5)}{2} + 5 \leq m \leq \frac{n(n-5)}{2} + 7 = C_{n-2}^2 + 4.$$

Note that the upper bound $C_{n-2}^2 + 4$ for the number of edges m is reached only for the graph $G = K_2 \vee (K_{n-4} + 2K_1)$. Moreover, a graph H, obtained from the graph G by removing only one edge, can have only one of the following degree sequences:

1) $(2, 2, \underbrace{(n-3), \ldots, (n-3)}_{n-4}, (n-2), (n-2))$, i.e. $H = 2K_1 \vee (K_{n-4} + 2K_1)$;

2) $(2, 2, (n-4), (n-4), \underbrace{(n-3), \ldots, (n-3)}_{n-6}, (n-1), (n-1))$, i.e. $H = K_2 \vee (2K_1 \vee K_{n-6}) + 2K_1)$;

3) $(2, 2, (n-4), \underbrace{(n-3), \ldots, (n-3)}_{n-5}, (n-2), (n-1))$.

Using the structure of the adjacency matrices of these graphs we show that for them the inequality $\rho(H) < n-3$ is valid. Hence, for the graph, obtained from the graph G by removing two edges, this inequality is valid as well according to the following statement:

Lemma 3 [1]. *Let G be a simple connected graph and H be its proper subgraph. Then $\rho(G) > \rho(H)$.*

For the cases $k = 3; 5; 6$ one can easily check that the spectral radius of G satisfies the inequality $\rho(H) < n-3$. For the case $k = 4$ the unique graphs that satisfy the inequality $\rho(H) \geq n-3$ are $5K_1 \vee K_4$ and $K_3 \vee (K_{1,4} + K_1)$.

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